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GROUPS WITH TRIVIAL VIRTUAL AUTOMORPHISM GROUP

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ABSTRACT

We answer a question of A. Lubotzky and A. Mann by constructing examples of infinite groups G such that every isomorphism $\alpha : H \to K$ between subgroups H and K having finite index in G coincides with the identity on some subgroup of finite index. The structure of such a group is very restricted; G must be virtually a 2-group with finite central derived subgroup and G/G' elementary abelian.

1. Definitions and introduction

We begin by giving some background to the question asked by A. Lubotzky and A. Mann [2]. Given a group G, they considered isomorphisms $\alpha : H \to K$, where H and K are subgroups of finite index in G. Given two such isomorphisms $\alpha_1 : H_1 \to K_1$ and $\alpha_2 : H_2 \to K_2$, they have a composition $\alpha_1\alpha_2$: $L \to M$, where $L = (K_1 \cap H_2)\alpha_1^{-1}$ and $M = (K_1 \cap H_2)\alpha_2$. This is the usual definition of composition of partial maps under which the set of such isomorphisms forms an inverse semigroup. To obtain a group we need to consider equivalence classes of such isomorphisms. Two isomorphisms $\alpha_1 : H_1 \to K_1$ and $\alpha_2 : H_2 \to K_2$ are said to be equivalent $(\alpha_1 \sim \alpha_2)$ if there is a subgroup L $(\leq H_1 \cap H_2)$ of finite index in G such that α_1 and α_2 coincide on L. It is clear that if $\alpha_1 \sim \beta_1$ and $\alpha_2 \sim \beta_2$, then $\alpha_1\alpha_2 \sim \beta_1\beta_2$.

An equivalence class $[\alpha]$ is called a virtual automorphism of G and the above composition induces a product of virtual automorphisms; $[\alpha][\beta]$ is defined to be

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$[\alpha\beta]$. With this definition it is a simple matter to see that the set of virtual automorphisms of G forms a group Vaut G. We understand that this group has also been considered by other authors under different names.

It is clear that if H is a subgroup of finite index in G, then Vaut $H \cong$ Vaut G. In particular, if G is finite then Vaut G is trivial. On the other hand, if G is infinite then Vaut G generally seems to have a rather richer structure than Aut G. For example, it is a simple exercise to show that Vaut $\mathbf{Z} \cong \mathbf{Q}^*$ and Vaut $\mathbf{Z}^n \cong \mathrm{GL}(n,\mathbf{Q})$. Lubotzky and Mann asked whether there is an infinite group G with Vaut $G = 1$.

We answer this question by giving two examples of infinite groups satisfying this condition. The reductions in the problem given in Section 2 show that if G is an infinite group with Vaut $G = 1$ then G contains a subgroup L of finite index such that L is a 2-group with L' finite and central, L/L' elementary abelian and Vaut $L = 1$. Thus, in constructing our examples, we need only consider 2-groups of this form. Our first example is of an extraspecial 2-group of cardinality $2^{\aleph_0}$ but has the disadvantage that our construction depends on the Continuum Hypothesis. Our second example does not require the Continuum Hypothesis, but is rather more complicated, having derived subgroup isomorphic to the 4-group. Again, this example has cardinality $2^{\aleph_0}$ and we have been unable to construct examples of other cardinalities. It does seem possible that the methods used here might yield examples of higher cardinalities, but the question of whether there are countably infinite groups G with Vaut $G = 1$ seems to be a much more difficult problem.

It is clear that if Vaut $G = 1$ then every automorphism α of G must coincide with the identity automorphism on a subgroup of finite index or, equivalently, $|G : C_G(\alpha)|$ is finite. Such automorphisms were called *bounded* by Zalesskiĭ [5] and *virtually trivial* by Menegazzo and Robinson [3]. In [3] groups of which every automorphism is virtually trivial, or *VTA-groups*, were considered. Examples of such groups were constructed and, more importantly for our purpose, a number of reduction theorems on the structure of *VTA*-groups were obtained. The results which we require may be summarized as follows.

PROPOSITION 1.1. (Menegazzo and Robinson [3]) *Let G be a VTA-group. Then*

(i) *G' is finite,*

(ii) *$G/Z(G)$ is centre-by-finite and has finite exponent,*

(iii) *$Z(G)$ is reduced and its primary components are finite,*

(iv) *if G is periodic, then $Z(G)$ is finite and G has finite exponent.*

If Vaut $G = 1$, then every subgroup of finite index in G is a *VTA*-group. Our reduction theorems are based on this fact together with the results given above.

2. Reduction theorems

LEMMA 2.1. *Let G be a nilpotent group of class two whose centre contains two subgroups A and B such that*

(a) $G' \leq A$,

(b) $A \cap B = 1$,

(c) $G/AB = Dr_{i \in I}\langle x_i AB \rangle$, *where* $\langle x_i \rangle \cap AB \leq B$.

Then G has an automorphism α fixing each element of A, inverting each element of B and inverting each x_i.

PROOF. If we order the index set I, then each element of G is uniquely expressible in the form

$$abx_{i_1}^{e_1} \cdots x_{i_m}^{e_m}$$

with $a \in A$, $b \in B$, $i_1 < \cdots < i_m$ and $1 \leq e_j \leq |\langle x_{i_j} AB \rangle|$.

We define the map $\alpha : G \to G$ by

$$(abx_{i_1}^{e_1} \cdots x_{i_m}^{e_m})\alpha = ab^{-1}x_{i_1}^{-e_1} \cdots x_{i_m}^{-e_m}.$$

A straightforward calculation verifies that α is an automorphism of G and it clearly satisfies the conditions described in the conclusion of the lemma.

THEOREM 2.2. *Let G be a group in which each subgroup of finite index is a VTA-group. If G is infinite then it contains a subgroup L of finite index such that*

(a) *L is nilpotent of class two,*

(b) *L is generated by elements of order two (and so is a 2-group),*

(c) *L' is finite.*

PROOF. The hypotheses of the theorem allow us to pass to subgroups of finite index. In a VTA-group G, $G/Z_2(G)$ is finite (Proposition 1.1(ii)) and so we may assume that G is nilpotent of class two. Also, G' is finite and $G/Z(G)$ has finite exponent (Propositions 1.1(i) and (ii)).

If G is periodic, let $A = G'$ and $B = 1$. By Proposition 1.1(iv) G, and hence also G/G', has finite exponent.

If G is not periodic, then the p-components of $Z(G)$ are finite (Proposition 1.1(iii)) and so G' is contained in a finite direct factor A of $Z(G)$. Thus $Z(G) = A \times B$ with $G' \leq A$. In both cases G/AB is abelian of finite exponent and so is a direct product of cyclic groups.

The finite abelian group A is the direct product of its Sylow p-subgroups A_p. Since G/B has finite exponent, each p-element a of A has finite p-height $k = k(a)$

in G modulo B; that is, $a \in G^{p^k}B - G^{p^{k+1}}B$. For each $a \in A_p$, choose an element $x(a) \in G$ such that $x(a)^{p^k} \in aB$ and consider

$$F = \langle x(a) : a \in A_p, p \in \pi(A)\rangle AB.$$

Since F is generated by the central subgroup AB and finitely many other elements, it follows that F/AB is finite and $G/C_G(F)$ is finite.

Since G/AB is abelian of finite exponent there is a subgroup K of finite index in G such that $K \leq C_G(F)$ and $K \cap F = AB$. Now K/AB is a direct product of finite cyclic groups and so we can choose elements $g_i \in K$, $i \in I$, such that g_iAB is a p-element and $K/AB = Dr_{i \in I}\langle g_iAB\rangle$.

Suppose that g_iAB has order p^l. If $g_i^{p^l} \in B$, put $x_i = g_i$. If $g_i^{p^l}B = aB$, where a is a non-trivial element of A, we may assume that a is a p-element (if a has order $p^m n$ with $p \nmid n$, then $\langle g_i^n AB\rangle = \langle g_iAB\rangle$ and $(g_i^n)^{p^l}B = a^nB$ with a^n of order p^m). In this case, there is a $k \geq l$ such that $a \in G^{p^k}B - G^{p^{k+1}}B$, and we put $x_i = g_ix(a)^{-p^{k-l}}$. Since g_i commutes with $x(a)$, $x_i^{p^l} = g_i^{p^l}x(a)^{-p^k} \in B$. Also, x_iAB has order p^l.

For, if $x_i^{p^{(l-1)}} \in AB$ then $g_i^{p^{(l-1)}}x(a)^{-p^{(k-1)}} \in AB$ and, since $K \cap F = AB$, we would have $g_i^{p^{(l-1)}} \in AB$ (and $x(a)^{-p^{(k-1)}} \in AB$).

Further, the elements x_iAB are independent. For, suppose $x_{i_1}^{e_1} \cdots x_{i_n}^{e_n} \in AB$; then $g_{i_1}^{e_1} \cdots g_{i_n}^{e_n} f \in AB$, where f is a product of powers of $x(a)$'s. But since $K \cap F = AB$, it follows that $g_{i_1}^{e_1} \cdots g_{i_n}^{e_n} \in AB$ and, by the independence of the g_iAB's, $g_{i_j}^{e_j} \in AB$. But x_iAB has the same order as g_iAB and so $x_{i_j}^{e_j} \in AB$.

Now let $H = \langle x_i : i \in I\rangle AB$ so that $H/AB = Dr_{i \in I}\langle x_iAB\rangle$ and $H' \leq G' \leq A$. By Lemma 2.1, H has an automorphism α which inverts the elements of B and inverts the elements x_i, $i \in I$.

But $FH = FK$ has finite index in G and, since F/AB is finite, it follows that $|G:H|$ is finite and so H is a VTA-group. In particular, the automorphism α is virtually trivial. But this can only occur if B is finite and all but finitely many of the elements x_i, $i \in I$, have order 2. Let $L = \langle x_i : x_i^2 = 1\rangle$. Then $|G:L|$ is finite and so L is a VTA-group generated by elements of order 2. Also, $L' \leq AB$ is finite.

In constructing our examples we concentrate on 2-groups of the form described in the theorem. Having constructed such a group, the main problem is of proving that every virtual automorphism is trivial.

3. The first example

In our first example we get around the difficulty of proving that every virtual automorphism of G is trivial by considering one of the consequences of G having a non-trivial virtual automorphism. This result provides a certain countably infinite subgroup of G and it is this which leads to our example having cardinality $2^{\aleph_0}$.

LEMMA 3.1. *Let G be a 2-group of class two such that G/G' is elementary abelian and G' is finite. Let $\alpha : H \to K$ be an isomorphism between subgroups of G such that $|H : C_H(\alpha)|$ is infinite. Then there is an infinite abelian subgroup A of H such that $A \cap A\alpha \leq G'$.*

PROOF. We show first that for any finite abelian subgroup F of H with $FG' \cap (F\alpha)G' = G'$ there is an element $x \in C_H(F) - FG'$ such that $\langle F,x\rangle G' \cap (\langle F,x\rangle\alpha)G' = G'$. Thus, let Y be the join of $F(F\alpha)G'$ and its pre-image under α; then $|Y|$ and $|H : C_H(F)|$ are finite, while $|H : C_H(\alpha)|$ is infinite, and so the set $\{[g,\alpha] : g \in C_H(F) - Y\}$ is infinite and cannot be contained in $F(F\alpha)G'$. If now $x \in C_H(F) - Y$ is such that $[x,\alpha] \notin F(F\alpha)G'$, it is easily seen that $\langle F,x\rangle G' \cap (\langle F,x\rangle\alpha)G' = G'$.

By the above remark we can inductively define elements $x_1, x_2, \ldots \in H$ and finite subgroups $A_n = \langle x_1, \ldots, x_n\rangle$ such that $x_n \in C_H(A_{n-1}) - A_{n-1}G'$ and $A_nG' \cap (A_n\alpha)G' = G'$. Let $A = \bigcup_{n=1}^{\infty} A_n = \langle x_1, x_2, \ldots\rangle$; then A is an infinite abelian subgroup of H. Let $y \in A \cap A\alpha$; then there is a least integer n such that $y \in A_n \cap A_n\alpha$, and it is clear from the definition of the A_n that $y \in G'$.

To construct a group G for which Vaut $G = 1$ we must therefore construct a 2-group G with G' finite such that for every isomorphism $\alpha : H \to K$ there is no infinite abelian subgroup $A \leq H$ such that $A \cap A\alpha = G'$. Alternatively, for each pair of infinite abelian subgroups A, B with $A \cap B = G'$ and each isomorphism $\alpha : A \to B$, α cannot be extended to an isomorphism $\alpha^* : H \to K$ with $|G : H|$ and $|G : K|$ finite.

The construction of an extraspecial 2-group with this property is based on the method used by A. Ehrenfeucht and V. Faber in [1] to construct infinite extraspecial groups in which each abelian subgroup has cardinality less than that of the whole group. This method requires the Continuum Hypothesis, but it should be noted that the Ehrenfeucht–Faber example has been constructed and generalized without using the Continuum Hypothesis by Shelah and Steprans [4]. It seems possible that their methods could be adapted to improve the example we construct here.

The results leading to the example are given in terms of an infinite cardinal $\boldsymbol{m}$, but because of the countable subgroup occurring in Lemma 3.1 we shall apply them in the case $\boldsymbol{m} = \aleph_0$.

LEMMA 3.2. ($2^{\boldsymbol{m}} = \boldsymbol{m}^+$) *Let V be a vector space of dimension $\boldsymbol{m}$ over* GF(p), *U a subspace of codimension* 1 *and $v \in V - U$.*

Let $\phi : U \times U \to \mathrm{GF}(p)$ be an alternate bilinear map and let $\mathfrak{A}$ be a family of linear bijections $\beta : X \to Y$ where X and Y are subspaces of U having dimension $\boldsymbol{m}$ such that $X \cap Y = 0$ and $|\mathfrak{A}| = \boldsymbol{m}$.

Then there is an alternate bilinear map $\phi' : V \times V \to \mathrm{GF}(p)$ extending ϕ and such that:

for each $(\beta, u_1, u_2) \in \mathfrak{A} \times U \times U$, there are elements $x, y \in X = \mathrm{Dom}\,\beta$ such that

(a) $\phi'(x, v) \neq \phi(x\beta, u_1) - \phi(x, u_2)$,

(b) $\phi'(x\beta, v) \neq \phi(x, u_2) - \phi(x\beta, u_1)$,

(c) $\phi'(y, v) - \phi'(y\beta, v) \neq \phi(y\beta, u_1) - \phi(y, u_2)$.

PROOF. Well order $\mathfrak{A} \times U \times U$ with order type μ, the least ordinal of cardinality $\boldsymbol{m}$, so that the elements are

$$(\beta_\lambda : X_\lambda \to Y_\lambda, u_1(\lambda), u_2(\lambda)), \qquad \lambda < \mu.$$

We can choose elements $x_\lambda, y_\lambda \in X_\lambda$ such that the elements $\{x_\lambda, y_\lambda, x_\lambda\beta_\lambda, y_\lambda\beta_\lambda : \lambda < \mu\}$ are linearly independent.

Extend the set $\{x_\lambda, y_\lambda, x_\lambda\beta_\lambda, y_\lambda\beta_\lambda : \lambda < \mu\}$ to a basis $\mathfrak{B}$ of U. Then $\mathfrak{B} \cup \{v\}$ is a basis of V.

The alternate bilinear map ϕ' can be specified by defining $\phi'(b,v)$, for all $b \in \mathfrak{B}$, and then putting $\phi'(v,b) = -\phi'(b,v)$ and $\phi'(v,v) = 0$. We define $\phi'(x_\lambda, v)$ to be different from $\phi(x_\lambda\beta_\lambda, u_1(\lambda)) - \phi(x_\lambda, u_2(\lambda))$, $\phi'(x_\lambda\beta_\lambda, v)$ to be different from $\phi(x_\lambda, u_2(\lambda)) - \phi(x_\lambda\beta_\lambda, u_1(\lambda))$, $\phi'(y_\lambda, v)$ and $\phi'(y_\lambda\beta_\lambda, v)$ so that $\phi'(y_\lambda, v) - \phi'(y_\lambda\beta_\lambda, v)$ is different from $\phi(y_\lambda\beta_\lambda, u_1(\lambda)) - \phi(y_\lambda, u_2(\lambda))$. For the remaining $b \in \mathfrak{B}$ we can define $\phi'(b,v)$ arbitrarily.

If $(\beta, u_1, u_2) \in \mathfrak{A} \times U \times U$, then there is a $\lambda < \mu$ such that $(\beta, u_1, u_2) = (\beta_\lambda, u_1(\lambda), u_2(\lambda))$. Then $x = x_\lambda$ and $y = y_\lambda$ are the required elements.

THEOREM 3.3. ($2^{\boldsymbol{m}} = \boldsymbol{m}^+$) *There is a symplectic space V of dimension $\boldsymbol{m}^+$ over* GF(p) *with alternate form ϕ and with the following property:*

If W is an $\boldsymbol{m}^+$-dimensional subspace of V and $\alpha : W \to V$ is an injective linear map such that W contains an $\boldsymbol{m}$-dimensional subspace X with $X \cap X\alpha = 0$, then α does not preserve ϕ.

PROOF. Let V have basis $\{v_\epsilon : \epsilon < \kappa\}$, where κ is the least ordinal of cardinality $\boldsymbol{m}^+$, and let $V_\epsilon = \langle v_\lambda : \lambda < \epsilon \rangle$.

Consider all pairs of $\boldsymbol{m}$-dimensional subspaces X, Y with $X \cap Y = 0$ and bijections $\beta: X \to Y$. There are $2^{\boldsymbol{m}} = \boldsymbol{m}^+$ pairs (X, Y) and for each pair there are $2^{\boldsymbol{m}} = \boldsymbol{m}^+$ bijections. So in total there are $2^{\boldsymbol{m}} = \boldsymbol{m}^+$ bijections of this type.

These may be ordered as $\beta_\epsilon : X_\epsilon \to Y_\epsilon$ with $\mu \leq \epsilon < \kappa$ in such a way that $X_\epsilon + Y_\epsilon \subseteq V_\epsilon$ (μ is the least ordinal of cardinality $\boldsymbol{m}$).

For each ϵ, $\mu \leq \epsilon < \kappa$, let $\mathfrak{A}_\epsilon = \{\beta_\lambda : X_\lambda \to Y_\lambda : \mu \leq \lambda \leq \epsilon\}$.

We define an alternate bilinear map ϕ on $V \times V$ as follows. Let ϕ_μ be any such map on $V_\mu \times V_\mu$ and then, inductively, extend ϕ_ϵ on $V_\epsilon \times V_\epsilon$ to $\phi_{\epsilon+1}$ on $V_{\epsilon+1} \times V_{\epsilon+1}$ using Lemma 3.2 with $U = V_\epsilon$, $v = v_\epsilon$, $\phi = \phi_\epsilon$ and $\mathfrak{A} = \mathfrak{A}_\epsilon$. This enables us to define $\phi : V \times V \to \mathrm{GF}(p)$ to coincide with ϕ_ϵ on each $V_\epsilon \times V_\epsilon$.

Now take an injective linear map $\alpha : W \to V$ with $\dim W = \boldsymbol{m}^+$ and with an $\boldsymbol{m}$-dimensional subspace $X \subseteq W$ such that $X \cap X\alpha = 0$. We show that α does not preserve the alternate bilinear map ϕ defined above.

The restriction of α to X occurs as some bijection $\beta_\epsilon : X_\epsilon \to Y_\epsilon$. Since $\dim W = \boldsymbol{m}^+$, W contains an element of the form $w + v_\delta$ with $w \in V_\delta$ and δ some ordinal greater than ϵ. We consider the image of $w + v_\delta$ under α; there are three possibilities:

(a) $(w + v_\delta)\alpha \in V_\delta$,

(b) $(w + v_\delta)\alpha = u + v_\gamma$ with $u \in V_\gamma$, some $\gamma > \delta$,

(c) $(w + v_\delta)\alpha = u + v_\delta$ with $u \in V_\delta$.

Case (a): Consider the extension from ϕ_δ to $\phi_{\delta+1}$. Applying Lemma 3.2(a) with $u_1 = (w + v_\delta)\alpha$ and $u_2 = w$, there is an $x \in X$ such that

$$\phi(x, v_\delta) \neq \phi(x\alpha, (w + v_\delta)\alpha) - \phi(x, w)$$

and so $\phi(x\alpha, (w + v_\delta)\alpha) \neq \phi(x, w + v_\delta)$.

Case (b): Consider the extension from ϕ_γ to $\phi_{\gamma+1}$. Applying Lemma 3.2(b) with $u_1 = u$ and $u_2 = w + v_\delta$, there is an $x \in X$ such that

$$\phi(x\alpha, v_\gamma) \neq \phi(x, w + v_\delta) - \phi(x\alpha, u)$$

and so $\phi(x\alpha, (w + v_\delta)\alpha) = \phi(x\alpha, u + v_\gamma) \neq \phi(x, w + v_\delta)$.

Case (c): Consider the extension from ϕ_δ to $\phi_{\delta+1}$. Applying Lemma 3.2(c) with $u_1 = u$ and $u_2 = w$, there is a $y \in X$ such that

$$\phi(y, v_\delta) - \phi(y\alpha, v_\delta) \neq \phi(y\alpha, u) - \phi(y, w)$$

so that $\phi(y\alpha, (w + v_\delta)\alpha) = \phi(y\alpha, u + v_\delta) \neq \phi(y, w + v_\delta)$.

In all three cases we have $x \in X$ such that $\phi(x\alpha, (w + v_\delta)\alpha) \neq \phi(x, w + v_\delta)$ and so α does not preserve the alternate form ϕ.

THEOREM 3.4. ($2^{\aleph_0} = \aleph_1$) *There is an extraspecial 2-group G of cardinality $2^{\aleph_0}$ such that every monomorphism $\alpha : H \to G$ with $|H| = 2^{\aleph_0}$ satisfies $|H : C_H(\alpha)|$ is finite. In particular,* Vaut $G = 1$.

PROOF: Let V be the symplectic space of dimension $2^{\aleph_0}$ over GF(2) given by Theorem 3.3. Define G to be the extraspecial 2-group generated by a central involution z and involutions x_ϵ, $\epsilon < 2^{\aleph_0}$, such that $[x_\delta, x_\epsilon] = z^{\phi(v_\delta, v_\epsilon)}$. Then G/G' can be identified with the symplectic space V. Given a monomorphism $\alpha : H \to G$, α induces an injective linear map α^* of the subspace HG'/G' into $G/G' = V$ which preserves the bilinear form ϕ.

If there is an $\alpha : H \to G$ with $|H : C_H(\alpha)|$ infinite then, by Lemma 3.1, there is an abelian subgroup A of H such that $|A| = \aleph_0$ and $A \cap A\alpha \leq G'$. Consider $AG'/G' \cap (A\alpha)G'/G'$; this is finite and so there is an infinite dimensional subspace $B \subseteq AG'/G'$ such that $B \cap B\alpha^* = 0$. By Theorem 3.3, α^* does not preserve the bilinear form ϕ and so we have a contradiction. Hence $|H : C_H(\alpha)|$ is finite for all monomorphisms $\alpha : H \to G$.

4. The second example

Our second construction also depends on considering G/G' as a vector space over GF(2) and will be obtained by taking G/G' to be formed from a vector space V and a certain subspace of its dual space V^*. We begin by constructing this subspace in Proposition 4.2. The results here could be given for vector spaces over GF(p) but for ease of calculation we only present them for GF(2).

LEMMA 4.1. *Let V be a vector space of dimension $\aleph_0$ over* GF(2) *and let T be a subspace of $V^* =$* Hom(V,GF(2)) *such that* $\dim T < 2^{\aleph_0}$. *Let $\sigma : V \to V$ be a linear map such that* codim $V\sigma$ *is finite and* $\dim V(\sigma - 1)$ *is infinite. Then*

(i) *there is an $f \in V^*$ such that $T + \langle f \rangle + \langle \sigma f \rangle = T \oplus \langle f \rangle \oplus \langle \sigma f \rangle$;*

(ii) *there is a subspace $S \subseteq V^*$ such that* $\dim S = \dim \sigma S = \aleph_0$ *and $T + S + \sigma S = T \oplus S \oplus \sigma S$.*

PROOF. (i) Consider the map $\sigma^* : V^* \to V^*$ defined by $f\sigma^* = \sigma f$, for all $f \in V^*$. Then Ker $\sigma^* = (V\sigma)^\perp$ has finite dimension. Also, $\mathrm{Ker}(\sigma - 1)^* = (V(\sigma - 1))^\perp$ has codimension equal to $\dim(V(\sigma - 1))^* = 2^{\aleph_0}$. If $R = T + \{f \in V^* : \sigma f \in T\} + \{f \in V^* : (\sigma - 1)f \in T\}$, then R is a subspace of V^* of codimension $2^{\aleph_0}$.

Therefore, there is an element $f \in V^* - R$. We have $f \notin T$, $\sigma f \notin T$ and $f + \sigma f \notin T$. Therefore, $T + \langle f \rangle + \langle \sigma f \rangle = T \oplus \langle f \rangle \oplus \langle \sigma f \rangle$.

(ii) By the above, we can choose an $f_0 \in V^*$ such that $T + \langle f_0 \rangle + \langle \sigma f_0 \rangle = T \oplus \langle f_0 \rangle \oplus \langle \sigma f_0 \rangle = T_1$, say. Suppose that we have chosen $f_0, \ldots, f_{n-1} \in V^*$ such that

$$\dim\langle f_0, \ldots, f_{n-1} \rangle = \dim\langle \sigma f_0, \ldots, \sigma f_{n-1} \rangle = n$$

and

$$\begin{aligned} T_n &= T + \langle f_0, \ldots, f_{n-1} \rangle + \langle \sigma f_0, \ldots, \sigma f_{n-1} \rangle \\ &= T \oplus \langle f_0, \ldots, f_{n-1} \rangle \oplus \langle \sigma f_0, \ldots, \sigma f_{n-1} \rangle. \end{aligned}$$

Then $\dim T_n < 2^{\aleph_0}$ and so, by (i), there is an $f_n \in V^*$ such that $T_n + \langle f_n \rangle + \langle \sigma f_n \rangle = T_n \oplus \langle f_n \rangle \oplus \langle \sigma f_n \rangle$. Hence

$$\dim\langle f_0, \ldots, f_n \rangle = \dim\langle \sigma f_0, \ldots, \sigma f_n \rangle = n + 1$$

and

$$\begin{aligned} T_{n+1} &= T + \langle f_0, \ldots, f_n \rangle + \langle \sigma f_0, \ldots, \sigma f_n \rangle \\ &= T \oplus \langle f_0, \ldots, f_n \rangle \oplus \langle \sigma f_0, \ldots, \sigma f_n \rangle. \end{aligned}$$

Having chosen the f_n inductively, for all n, we now define $S = \langle f_0, f_1, \ldots \rangle$; this clearly satisfies the required conditions.

PROPOSITION 4.2. *Let V be a vector space of dimension $\aleph_0$ over* GF(2) *and let R be a subspace of V^* of dimension $\aleph_0$.*

Then there is a subspace T of V^* such that $\dim T = 2^{\aleph_0}$ *and* $T = R \oplus S$ *with* $\dim S^\perp$ *finite satisfying:*

for each linear map $\sigma : V \to V$ with $\dim(\operatorname{Ker}\sigma)$ *finite,* $\operatorname{codim} V\sigma$ *finite and* $\dim V(\sigma - 1)$ *infinite, $T \cap \sigma T$ has infinite codimension in T.*

PROOF: The set of all linear maps $\sigma : V \to V$ such that $\dim(\operatorname{Ker} \sigma)$ is finite, $\operatorname{codim} V\sigma$ is finite and $\dim V(\sigma - 1)$ is infinite has cardinality $2^{\aleph_0}$ and so may be well-ordered as $\{\sigma_\epsilon : \epsilon < 2^{\aleph_0}\}$. Given $\epsilon < 2^{\aleph_0}$, suppose that for every $\gamma < \epsilon$ there is a subspace S_γ of V^* such that

$$\dim S_\gamma = \dim \sigma_\gamma S_\gamma = \aleph_0$$

and

$$T_\epsilon = \langle R, S_\gamma, \sigma_\gamma S_\gamma : \gamma < \epsilon \rangle = R \oplus \textstyle\bigoplus_{\gamma<\epsilon} S_\gamma \oplus \bigoplus_{\gamma<\epsilon} \sigma_\gamma S_\gamma.$$

Then $\dim T_\epsilon = |\epsilon|\ \aleph_0 < 2^{\aleph_0}$.

By Lemma 4.1(ii), there is a subspace $S_\epsilon \subseteq V^*$ with

$$\dim S_\epsilon = \dim \sigma_\epsilon S_\epsilon = \aleph_0$$

and

$$T_\epsilon + S_\epsilon + \sigma_\epsilon S_\epsilon = T_\epsilon \oplus S_\epsilon \oplus \sigma_\epsilon S_\epsilon .$$

Then

$$T_{\epsilon+1} = \langle R, S_\gamma, \sigma_\gamma S_\gamma : \gamma \leq \epsilon \rangle = R \oplus \bigoplus_{\gamma\leq\epsilon} S_\gamma \oplus \bigoplus_{\gamma\leq\epsilon} \sigma_\gamma S_\gamma .$$

We define $S = \bigoplus_{\epsilon<2^{\aleph_0}} S_\epsilon$ and $T = R \oplus S$. It is clear that $\dim T = \dim S = 2^{\aleph_0}$.

If $\sigma : V \to V$ is any linear map with $\dim(\operatorname{Ker}\sigma)$ finite, codim $V\sigma$ finite and $\dim V(\sigma - 1)$ infinite, then there is a linear map τ satisfying the same conditions and such that $\sigma\tau$ is the identity map on a subspace of finite codimension in V. Now $\tau = \sigma_\epsilon$ for some $\epsilon < 2^{\aleph_0}$ and $\tau S_\epsilon \cap T = \sigma_\epsilon S_\epsilon \cap (R + S) = 0$. Then $\tau T + T/T \supseteq \tau S_\epsilon + T/T \cong \tau S_\epsilon$ and so $\dim(\tau T + T/T)$ is infinite. But then $\dim(\sigma(\tau T + T)/\sigma T)$ is infinite and hence $\dim(T/T \cap \sigma T) = \dim(T + \sigma T/\sigma T)$ is infinite.

If $\dim S^\perp$ is infinite then there is a σ_ϵ such that $V(\sigma_\epsilon - 1) = S^\perp$. But for $f \in S_\epsilon \subseteq S$ and $v \in V$, we have $v(\sigma_\epsilon - 1)f = 0$ and so $\sigma_\epsilon f = f$ contrary to $\sigma_\epsilon S_\epsilon \cap S_\epsilon = 0$.

Corollary 4.3. *Let V be a vector space of dimension $\aleph_0$ over* GF(2), *and suppose T is a subspace of V^* satisfying the conclusion of Proposition* 4.2. *Let X, Y be subspaces of V of finite codimension, let $\tau : X \to Y$ be an isomorphism with $\dim X(\tau - 1)$ infinite and let $\rho : V^* \to X^*$ be the restriction map. Then $T\rho \cap \tau T$ has infinite codimension in $T\rho$.*

Proof. The map τ can be extended to a map $\sigma : V \to V$ simply by defining σ to be 0 on some complement to X. Then $\dim(\operatorname{Ker}\sigma)$ is finite, codim $V\sigma$ is finite and $\dim V(\sigma - 1)$ is infinite. By Proposition 4.2, the subspace T of V^* is such that $\dim T = 2^{\aleph_0}$ and $\sigma T \cap T$ has infinite codimension in T. But $\tau T = (\sigma T)\rho$ and so $T\rho/T\rho \cap \tau T = T\rho/T\rho \cap \sigma T\rho \cong T + \sigma T + X^\perp/\sigma T + X^\perp$.

But $X^\perp$ has finite dimension while $\dim(T + \sigma T/\sigma T)$ is infinite, and so $T\rho/T\rho \cap \tau T$ has infinite dimension.

Theorem 4.4. *There is an infinite 2-group G with G/G' elementary abelian and G' a central subgroup isomorphic to $C_2 \times C_2$ such that every isomorphism $\alpha : H \to K$ between subgroups of finite index in G coincides with the identity on a subgroup of finite index in H.*

PROOF. Let E and F be extraspecial 2-groups of cardinalities $|E| = \aleph_0$, $|F| = 2^{\aleph_0}$ and centres $Z(E) = E' = \langle a \rangle$, $Z(F) = F' = \langle b \rangle$.

Let V be a vector space of dimension $\aleph_0$ over GF(2) and let $\phi : E \to V$ be a surjective group homomorphism with Ker $\phi = E'$. For each $e_0 \in E$ the map $e \mapsto [e_0, e]$ induces a linear map $e_0\chi \in V^*$. Let $R = E\chi$, a subspace of V^* of dimension $\aleph_0$. By Proposition 4.2, there are subspaces S and $T = R \oplus S$ of V^* such that $\dim T = 2^{\aleph_0}$, $\dim S^\perp$ is finite and, for each linear map $\sigma : V \to V$ with $\dim(\operatorname{Ker} \sigma)$ finite, codim $V\sigma$ finite and $\dim V(\sigma - 1)$ infinite, $T \cap \sigma T$ has infinite codimension in T.

Let $\psi : F \to S$ be a surjective group homomorphism with Ker $\psi = F'$. For each $(e,f) \in E \times F$, define

$$r(e,f) = (e\phi)(f\psi) \in \mathrm{GF}(2).$$

Then we have an action of F on E

$$e^f = ea^{r(e,f)}$$

and this action has kernel F'. We form the split extension G of E by F with this action. It is clear that $G' = \langle a \rangle \times \langle b \rangle = Z(G)$, $C_G(G/\langle a \rangle) = E \times \langle b \rangle$ and $C_G(G/\langle b \rangle) = C_G(G/\langle ab \rangle) = G'$. In particular, $\langle a \rangle$ is a characteristic subgroup of G.

If E_1 is a subgroup of finite index in E, then $C_G(E_1)$ is finite. For $E_1\phi$ has finite codimension in V and so $\dim(E_1\phi)^\perp$ is finite; hence $C_G(E_1/\langle a \rangle) = EF_0$ for some finite subgroup F_0 of F. Clearly, $C_G(E_1) \le EF_0$ and so if $C_G(E_1)$ were infinite it would follow that $C_E(E_1)$ is infinite, which is clearly not the case. In a similar way we see that $C_G(F_1)$ is finite for each subgroup F_1 of finite index in F.

Let H be any subgroup of finite index in G; then $H \ge G'$ and $C_H(H/\langle b \rangle) \le C_H(E \cap H)$ and $C_H(H/\langle ab \rangle) \le C_H(E \cap H)$ so that $C_H(H/\langle b \rangle)$ and $C_H(H/\langle ab \rangle)$ are both finite. On the other hand, $E \cap H \le C_H(H/\langle a \rangle)$ and if $h = ef \in C_H(H/\langle a \rangle)$ then, since E centralizes $G/\langle a \rangle$, we have $[f, F \cap H] \le F \cap \langle a \rangle = 1$ so that $C_H(H/\langle a \rangle) \le EC_G(F \cap H)$ and

$$(*) \qquad |C_H(H/\langle a \rangle) : E \cap H| \text{ is finite.}$$

Now let $\alpha : H \to K$ be an isomorphism with $|G:H|$ and $|G:K|$ finite. We suppose that $|H : C_H(\alpha)|$ is infinite and obtain a contradiction.

Suppose first that $E_1 = E \cap C_H(\alpha)$ has finite index in E. Then $E_1 \ge E'$, $E_1 \lhd G$ and, for each $e \in E_1$ and $f \in F \cap H$, we have

$$e^f = (e^f)\alpha = (e\alpha)^{(f\alpha)} = e^{(f\alpha)}.$$

Thus $[f,\alpha] \in C_G(E_1)$, which is a finite subgroup. Since $[E \cap H, \alpha]$ and $[F \cap H, \alpha]$ are finite, it follows that $[H,\alpha]$ is finite and hence $|H : C_H(\alpha)|$ is finite, contrary to hypothesis.

Therefore we may assume that $|E : E \cap C_H(\alpha)|$ is infinite. Then $a\alpha = a$ and $C_H(H/\langle a\rangle)\alpha = C_{H\alpha}(H\alpha/\langle a\rangle)$; also, $|E : E \cap H|$ and $|E : E \cap H\alpha|$ are finite. By (*), both $|C_H(H/\langle a\rangle) : E \cap H|$ and $|C_{H\alpha}(H\alpha/\langle a\rangle) : E \cap H\alpha|$ are finite; taking images under α and α^{-1}, respectively, we find that $|C_{H\alpha}(H\alpha/\langle a\rangle) : (E \cap H)\alpha|$ and $|C_H(H/\langle a\rangle) : (E \cap H\alpha)\alpha^{-1}|$ are finite as well. Therefore, if we set $L = (E \cap H) \cap (E \cap H\alpha)\alpha^{-1}$, both L and $L\alpha = (E \cap H)\alpha \cap (E \cap H\alpha)$ have finite index in E. Thus α induces an isomorphism between subgroups L and $L\alpha$ of finite index in E. Clearly, $|L : C_L(\alpha)|$ is infinite.

For each $g \in G$, the map $e \mapsto [e,g]$ induces an element $g\xi \in V^*$. If $g \in E$ then $g\xi = g\chi$ and if $g \in F$ then $g\xi = g\psi$. Thus $G\xi = E\chi + F\psi = R + S = T$.

For each $h \in H$, $h\xi$ and $h\alpha\xi$ are elements of T. Let $X = L\phi$; then X is a subspace of finite codimension in V and α induces an isomorphism τ from X to $Y = (L\alpha)\phi$. Moreover, $\dim X(\tau - 1)$ is infinite, for otherwise we would have $|L : C_L(\alpha)|$ finite. For each $l \in L$, $[l,h] \in \langle a\rangle$; hence $[l,h] = [l,h]\alpha = [l\alpha, h\alpha]$ and so $h\xi$ coincides with $\tau h\alpha\xi$ on X. If $\rho : V^* \to X^*$ is the restriction map, then we have $H\xi\rho \leq T\rho \cap \tau T$. By Corollary 4.3, this has infinite codimension in $T\rho = G\xi\rho$ and this contradicts $|G : H|$ being finite.

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